**5398**: **Proposed by D. M. Bătinetu-Giurgiu**, **Bucharest**, **Romania and Neculai Stanciu**, **"George Emil Palade" General School**, **Buzău**, **Romania**

If $(2n-1)!! = 1\cdot 3\cdot 5\ldots(2n-1)$, then evaluate

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!(2n-1)!!}}{n}\right)$$

**Solution by Arkady Alt** , **San Jose** ,**California**, **USA**.

Since $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \Rightarrow \frac{\sqrt[2n]{(2n)!}}{2n} = \frac{1}{e}$ and $n!(2n-1)!! = \frac{(2n)!}{2^n} \Leftrightarrow$

$\sqrt[n]{n!(2n-1)!!} = \frac{\sqrt[n]{(2n)!}}{2} \Leftrightarrow \frac{\sqrt[n]{n!(2n-1)!!}}{n} = \frac{\sqrt[n]{(2n)!}}{2n} = \left(\frac{\sqrt[2n]{(2n)!}}{2n}\right)^2\cdot 2n$ then

$\frac{\sqrt[n]{n!(2n-1)!!}}{n} \sim \frac{2n}{e^2}$ (notation $a_n \sim b_n$ ($a_n$ and $b_n$asymptotically equivalent)

mean $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$),

that is $\lim_{n\to\infty}\frac{\sqrt[n]{n!(2n-1)!!}}{2n^2} = \frac{1}{e^2}$. Let $c_n := \frac{\sqrt[n]{n!(2n-1)!!}}{2n^2}$ and $\alpha_n := \ln\left(\frac{c_{n+1}}{c_n}\cdot\frac{n}{n+1}\right)$.

Then $\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!(2n-1)!!}}{n} = \frac{\sqrt[n]{n!(2n-1)!!}}{n}\left(\frac{n\sqrt[n+1]{(n+1)!(2n+1)!!}}{(n+1)\sqrt[n]{n!(2n-1)!!}} - 1\right)$

$nc_n\left(\frac{c_{n+1}}{c_n}\cdot\frac{n}{n+1} - 1\right) = nc_n(e^{\alpha_n}-1) = c_n\cdot\frac{e^{\alpha_n}-1}{\alpha_n}\cdot n\alpha_n.$

Since $\lim_{n\to\infty} c_n = \frac{1}{e^2}$ and $\lim_{n\to\infty}\frac{c_{n+1}}{c_n}\cdot\frac{n}{n+1} = 1 \Leftrightarrow \lim_{n\to\infty}\alpha_n = 0$ then $\lim_{n\to\infty}\frac{e^{\alpha_n}-1}{\alpha_n} = 1$ and

$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!(2n-1)!!}}{n}\right) = \frac{1}{e^2}\lim_{n\to\infty} n\alpha_n.$

We have $n\alpha_n = \ln\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!(2n+1)!!}}\cdot\frac{(n+1)!(2n+1)!!}{n!(2n-1)!!}\right) =$

$\ln\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{(n+1)(2n+1)}{\sqrt[n+1]{(n+1)!(2n+1)!!}}\right) = -\ln\left(1+\frac{1}{n}\right)^n + \ln\frac{1}{c_{n+1}} + \ln\frac{2n+1}{n+1}$

and, therefore, $\lim_{n\to\infty} n\alpha_n = -\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n - \lim_{n\to\infty}\ln c_{n+1} + \lim_{n\to\infty}\ln\frac{2n+1}{n+1} =$

$-1 + 2 + \ln 2 = \ln 2 - 1.$

Thus, $\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!(2n-1)!!}}{n}\right) = \frac{\ln 2 - 1}{e^2}.$